TRANSITIVE PERMUTATION GROUPS OF PRIME DEGREE, III: CHARACTER-THEORETIC OBSERVATIONS

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1. Introductory remarks

As in my earlier paper [**13**], I intend to report results from a study of a transitive group G of permutations of a set Ω where $|\Omega| = p$ and p is a prime number. Since if G is soluble its structure is well understood, it will be assumed throughout that G is insoluble. In the previous work I used representation theory to show that if a Sylow p-normaliser $N(P)$ has even order then G must be 3-fold transitive. Those ideas can be extended a little to give more information about the characters of G. What one achieves is the proof of the results numbered 3.1, 4.1, 4.5, 4.6, 4.7, and 5.1 below, which, though rather technical and less strong than I would like, can nevertheless be used to give a little more information about the way in which G must act on Ω. Here, as samples, are two such theorems.

THEOREM 4.2. *If $N(P)$ has even order then G is a little generously 3-fold transitive.*

This means (see [**14**]) that, given any four points $\alpha, \beta, \gamma, \delta$ in Ω, there is a permutation $g \in G$ whose cycle decomposition has the form $(\alpha\beta)(\gamma\delta)\ldots$.

THEOREM 5.2. *If $|N(P)| = tp$ where t is odd then*

(i) *$G_{\alpha\beta}$ has at most $(p-1)/t$ orbits in $\Omega \setminus \{\alpha, \beta\}$, and*

(ii) *$G_{\{\alpha\beta\}}$ has at most $1 + (p-1)/2t$ orbits in $\Omega \setminus \{\alpha, \beta\}$.*

2. The characters of G according to R. Brauer and W. Feit

My main tool is the description of the characters of G originally given in the famous paper ([**4**]) of Richard Brauer. Except where explicitly stated otherwise my only assumptions in this section will be (i) that if $X \lhd G$ and $p \nmid |X|$ then $X = 1$, and (ii) that if P denotes a Sylow p-subgroup of G then $|P| = p$ and $C(P) = P$. Certainly a transitive permutation group of prime degree p does satisfy these conditions. The extra generality is not great, and Brauer's work offers vastly more, but it will be quite sufficient for our purposes. With those assumptions it is not hard to see that there is a unique minimal non-trivial normal subgroup G_0 in G; G_0 is simple, it is generated by the conjugates of P, and, since $G = N(P)G_0$, the factor group G/G_0 is cyclic and its order divides $p-1$.

The normaliser $N(P)$ is a metacyclic group PQ generated by a and b; $\langle a\rangle = P$, $\langle b\rangle = Q$, $|Q| = t$, and $b^{-1}ab = a^{\beta}$ where β is a primitive tth root of 1 modulo p.

According to Brauer's theory some of the ordinary absolutely irreducible characters of G lie in the principal p-block $B_0(G)$. If χ is any irreducible character not in $B_0(G)$ then χ is said to have defect 0, and this can be recognized from the fact that p divides $\chi(1)$ (which is the degree of χ) or that $\chi(x) = 0$ for any non-trivial element x of P. It is the principal p-block which seems to be of most use for my purposes. It contains $t+(p-1)/t$ characters $\chi_0, \chi_1, \ldots, \chi_{t-1}, \theta_1, \ldots, \theta_{(p-1)/t}$, which have the following properties. For certain $\varepsilon_0, \ldots, \varepsilon_t$ such that $\varepsilon_i = \pm 1$,

$$\chi_i(1) \equiv \varepsilon_i \mod p,$$
$$\chi_i(x) = \varepsilon_i \quad (\text{for all } x \in P\setminus\{1\})$$

if $0 \leqslant i \leqslant t-1$, and

$$\theta_j(1) \equiv -\varepsilon_t t \mod p,$$
$$\theta_j(a^{\nu}) = -\varepsilon_t(\omega_j^{\nu} + \omega_j^{\nu\beta} + \ldots + \omega_j^{\nu\beta^{t-1}})$$

if $1 \leqslant j \leqslant (p-1)/t$, where $\omega_j = \omega^{\rho_j}$, ω is a primitive complex pth root of 1, and $\rho_1, \ldots, \rho_{(p-1)/t}$ are integers which form a complete set of coset representatives for $\{1, \beta, \ldots, \beta^{t-1}\}$ in the multiplicative group of non-zero integers modulo p. These characters $\theta_1, \ldots, \theta_{(p-1)/t}$ are called the exceptional characters in $B_0(G)$. It is convenient to define $\chi_t := \theta_1 + \ldots + \theta_{(p-1)/t}$, so that

$$\chi_t(1) \equiv \varepsilon_t \mod p \quad \text{and} \quad \chi_t(x) = \varepsilon_t \quad (x \in P\setminus\{1\}).$$

If $t = p-1$ then more delicate criteria are necessary to distinguish the exceptional character from the other characters in $B_0(G)$.

Let K be a p-adic number field which is a splitting field for G, let R be its local ring of integers with maximal ideal $\mathfrak{p}$, and let $\bar{K}$ denote the residue class ring $R/\mathfrak{p}$. Then $\bar{K}$ will be a finite field of characteristic p and a splitting field for G (cf. [7], p. 592). If M is an RG-module then we put $\bar{M} := \bar{K} \otimes M = M/\mathfrak{p}M$. Since K has characteristic 0 any ordinary irreducible character χ of G may be read as the character of a KG-module; and, since R is a principal ideal domain, χ can then be realized as the character of an RG-module (cf. [7], p. 496). If the regular representation module RG_{RG} is split as a direct sum of indecomposable RG-submodules, known simply as projective indecomposables, then some of the summands afford irreducible characters χ of G; in this case χ has defect 0, and if M

is one of these summands then $K \otimes M$ is irreducible as a KG-module and $\bar{M}$ is irreducible as a $\bar{K}G$-module. The remaining projective indecomposables are in the principal p-block $B_0(G)$. Up to isomorphism there are just t of these, $U_0, \ldots, U_{t-1}$, and if τ_i is the character which U_i affords, then for suitable j, k

$$\tau_i = \chi_j + \chi_k.$$

The decompositions of these projective indecomposable characters can be neatly exhibited by means of a certain graph associated with $B_0(G)$, which turns out to be a connected tree. The $t+1$ nodes (vertices) of this so-called Brauer tree are labelled with $\chi_0, \ldots, \chi_t$, and its t edges are labelled with $\tau_0, \ldots, \tau_{t-1}$. The edge τ_i is incident with vertices χ_j, χ_k if and only if $\tau_i = \chi_j + \chi_k$.

To each projective indecomposable U_i there corresponds an irreducible $\bar{K}G$-module F_i defined as $\bar{U}_i/\mathrm{rad}\,\bar{U}_i$, and $F_0, \ldots, F_{t-1}$ is a complete set of non-isomorphic irreducible $\bar{K}G$-modules in the principal p-block $B_0(G)$. Let X_j be an RG-module which affords the character χ_j $(0 \leqslant j \leqslant t-1)$ or one of the characters θ_l (if $j = t$). Then $\chi_j + \chi_k = \tau_i$ if and only if $\bar{X}_j$ and $\bar{X}_k$ both have F_i as a composition factor. In terms of the Brauer tree this means that we could label the edges with the modular irreducibles $F_0, \ldots, F_{t-1}$ and specify that edge F_i is incident with vertex χ_j if and only if F_i is a composition factor of $\bar{X}_j$. The composition factors appear in the modules $\bar{X}_j$ always with multiplicity 1; and the degree of F_i is the multiplicity of U_i as a summand of RG_{RG}. Consequently the character χ_j represents an end-node (a node which lies on one and only one edge) of the Brauer tree if and only if $\bar{X}_j$ is irreducible.

We can specify that the numbering of characters was so chosen that χ_0 is the principal character, that is, $\chi_0 = 1$, and $\tau_0 = \chi_0 + \chi_1$. (This may introduce ambiguity by making our original χ_t equal to χ_1. However, the ambiguity is easily resolved, and it does not arise for the permutation groups which are our main interest here.) Suppose that $|G : G_0| = s$ where G_0 is the derived group of G, the unique non-trivial minimal normal subgroup (provided that $t \neq 1$, that is, that $G > P$). Then G has s different linear characters $\lambda_1, \lambda_2, \ldots, \lambda_s$ (where $\lambda_1 = 1 = \chi_0$), which form a cyclic group Λ under multiplication. If $L_1, \ldots, L_s$ are the corresponding 1-dimensional RG-modules then $\bar{L}_1, \ldots, \bar{L}_s$ are the s different 1-dimensional $\bar{K}G$-modules. If U is a projective indecomposable RG-module then so is $U \otimes L_i$ and therefore the map $\chi_j \mapsto \chi_j\lambda_i$, $\tau_j \mapsto \tau_j\lambda_i$ is an automorphism of the Brauer tree. In this way the group Λ acts on the Brauer tree as a group of automorphisms. The vertex χ_t is fixed by every member of Λ (if $t = p-1$ while $s > 1$ then we can take this as the property which defines the exceptional character χ_t). Apart from this the action of Λ is semi-regular: if $j \neq t$ and $i \neq 1$ then $\chi_j\lambda_i \neq \chi_j$. Let $V_i := U_1 \otimes L_i$ and let $\mu_i := \chi_1\lambda_i$ so that $V_1, \ldots, V_s$ are the projective indecomposables of G whose characters $\tau_0\lambda_i = \lambda_i + \mu_i$ have linear constituents. The characters $\mu_1, \ldots, \mu_s$ will either be distinct irreducible characters in $B_0(G)$ or they will all coincide in χ_t.

From the powerful theorem of Walter Feit ([**8**]; see also Blau's paper [**2**]) one can obtain information about the positions occupied in the Brauer tree by characters of relatively small degree.

LEMMA 2.1. *Suppose that G is insoluble, and that $G \not\cong \mathrm{PSL}(2,p)$ and $G \not\cong \mathrm{PGL}(2,p)$.*

(i) $\theta_i(1) \neq t$.

(ii) *If χ is an irreducible character of degree $p-1$ in $B_0(G)$ then either χ is an end-node, or $\chi = \mu_j$ for some j; in this latter case χ has valency 2 in the Brauer tree.*

(iii) *If χ is an irreducible character of degree $p+1$ then χ labels an end-node in the Brauer tree.*

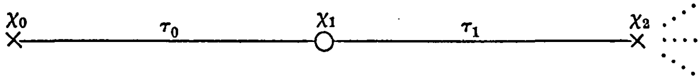
Part (i) is an old theorem of Brauer ([**5**]), which has been considerably extended by Feit ([**9**]). The proof of the whole lemma is an easy, and quite familiar, application of Theorem 1 of [**8**] (see, for example, [**13**], p. 207). By part (ii), if χ is an irreducible character of degree $p-1$ then there is exactly one projective indecomposable $\tau = \chi + \varphi$ in which φ is non-linear. I shall refer to φ as the non-linear mate of χ, and to τ as its inward link.

3. Characters of permutation groups

We return now to the assumption that G is insoluble and a transitive permutation group of degree p on Ω. It is a well-known theorem of Galois, whose proof was first completed by Jordan ([**12**]; cf. [**10**], p. 214), that if $p \geqslant 5$ then $\mathrm{PSL}(2,p)$ can be represented as a transitive group of degree p if and only if p is 5, 7, or 11. Therefore if $p > 11$ then $G \not\cong \mathrm{PSL}(2,p)$ and $G \not\cong \mathrm{PGL}(2,p)$, and we are well placed to apply Lemma 2.1.

The module $R\Omega$ is certainly indecomposable and projective, and as it has χ_0 as a constituent it must be U_0. Therefore χ_1 has degree $p-1$, and $\tau_0 = 1 + \chi_1$ is the permutation character of G. By Burnside's famous theorem ([**10**], p. 609), or from Lemma 2.1(i) (cf. [**6**], pp. 64, 65), G is doubly transitive, χ_1 is irreducible, and by Lemma 2.1(ii) there is just one other projective indecomposable character τ_1 having χ_1 as a constituent (this is true also if G is $\mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$). If $\tau_1 = \chi_1 + \chi_2$

then we have the following fragment of the Brauer tree:



Let $\Omega^{(2)}$, $\Omega^{\{2\}}$ denote the G-spaces of ordered and unordered pairs of distinct elements of Ω (as in [**14**], not as in [**13**]); let $X := R\Omega^{(2)}$, $Y := R\Omega^{\{2\}}$, and let Z be the kernel of the natural homomorphism of X onto Y. Then $X \simeq Y \oplus Z$ and each of X, Y, Z is projective. The map from $\Omega^{\{2\}}$ to $R\Omega$ described by $\{\omega_1, \omega_2\} \mapsto \omega_1 + \omega_2$ is easily seen to extend to an RG-morphism $f\colon Y \to R\Omega$. Moreover, since $2\omega_1 = (\omega_1 + \omega_2) - (\omega_2 + \omega_3) + (\omega_3 + \omega_1)$, and since 2 is invertible in R, f is surjective. Since $R\Omega$ is projective it follows that $Y \simeq Y_1 \oplus R\Omega$ where Y_1 is the kernel of f. Being a summand of a projective module Y_1 is projective (in [**13**], p. 206, this was deduced by less elementary means). The characters ξ, η, η_1, ζ afforded by X, Y, Y_1, Z can be expressed thus:

$$\begin{aligned}
\xi &= 1 + 2\chi^{(p-1,1)} + \chi^{(p-2,2)} + \chi^{(p-2,1^2)},\\
\eta &= 1 + \chi^{(p-1,1)} + \chi^{(p-2,2)},\\
\eta_1 &= \chi^{(p-2,2)},\\
\zeta &= \chi^{(p-1,1)} + \chi^{(p-2,1^2)}.
\end{aligned}$$

Here if (λ) is a partition of p then $\chi^{(\lambda)}$ denotes the restriction to G of the character $\chi^{(\lambda)}$ of the symmetric group S_p. We have that

$$\chi^{(p-1,1)} = \chi_1 = a_1 - 1,$$

and

$$\begin{aligned}
\chi^{(p-2,2)} &= \tfrac{1}{2}a_1(a_1 - 3) + a_2,\\
\chi^{(p-2,1^2)} &= \tfrac{1}{2}(a_1 - 1)(a_1 - 2) - a_2,
\end{aligned}$$

where $a_r(g)$ is the number of r-cycles in the decomposition of g as a product of disjoint cycles. Since ξ, η, η_1, and ζ are the characters of projective RG-modules they can be expressed as sums of the characters τ_i and irreducible characters of defect 0. In particular since χ_1 is a constituent of ζ, while χ_0 is not, it follows that $\tau_1 \subseteq \zeta$ and therefore that $\chi_2 \subseteq \chi^{(p-2,1^2)}$.

Our aim will be to find out as much as possible about the decomposition of η and ζ, or equivalently of $\chi^{(p-2,2)}$ and $\chi^{(p-2,1^2)}$, as sums of irreducible characters of G. The following lemma disposes of the possibility of linear constituents.

LEMMA 3.1. *If λ is a linear character of G then*

$$\langle \lambda, \chi^{(p-2,2)} \rangle_G = \langle \lambda, \chi^{(p-2,1^2)} \rangle_G = 0.$$

Proof. Since the derived group G_0 is simple it is 2-transitive on Ω, and hence it is transitive on $\Omega^{(2)}$. Therefore $\langle 1, \xi - 1\rangle_{G_0} = 0$, and since $\lambda_{G_0} = 1$ it follows that $\langle \lambda, \xi - 1\rangle_{G_0} = 0$. Therefore $\langle \lambda, \chi^{(p-2,2)}\rangle_G = \langle \lambda, \chi^{(p-2,1^2)}\rangle_G = 0$.

LEMMA 3.2. *Suppose that t is even and put $b_0 := b^{t/2}$, so that b_0 is an involution in $N(P)$. Let λ be a linear character of $N(P)$.*

(i) *If $\lambda(b_0) = 1$ then*

$$\langle \lambda, \eta\rangle_{N(P)} = (p-1)/t \quad \textit{and} \quad \langle \lambda, \zeta\rangle_{N(P)} = 0.$$

(ii) *If $\lambda(b_0) = -1$ then*

$$\langle \lambda, \eta\rangle_{N(P)} = 0 \quad \textit{and} \quad \langle \lambda, \zeta\rangle_{N(P)} = (p-1)/t.$$

LEMMA 3.3. *If t is odd and λ is a linear character of $N(P)$ then*

$$\langle \lambda, \eta\rangle_{N(P)} = \langle \lambda, \zeta\rangle_{N(P)} = (p-1)/2t.$$

The proofs of these lemmas are omitted because they are straightforward calculations with the known values of the characters concerned.

LEMMA 3.4. *If χ is an irreducible character of G, then the restriction $\chi_{N(P)}$ has a linear constituent unless $\chi(1) = p-1$.*

Proof. Certainly $\chi_{N(P)}$ has a linear constituent if and only if χ_P contains the principal character of P. Suppose first that $\chi(1) = kp+\varepsilon$ where ε is 0 (if $\chi \notin B_0(G)$), 1, or -1. Then

$$\langle \chi, 1\rangle_P = p^{-1} \sum_{g\in P} \chi(g) = p^{-1}\{(kp+\varepsilon)+(p-1)\varepsilon\} = k+\varepsilon.$$

This cannot be 0 unless $k = 1$ and $\varepsilon = -1$, in which case χ has degree $p-1$. Only the exceptional characters are not covered by this argument. If χ is an exceptional character and its degree is $kp-\varepsilon t$, with values as described on p. 483, then we find that $\langle \chi, 1\rangle_P = k$. If this is 0 then ε_t must be -1 and the degree of χ is t. But then, by Lemma 2.1(i) (see also [5]) G is one of the groups PSL$(2,p)$ or PGL$(2,p)$ which we have excluded.

4. The decomposition of the permutation characters when t is even

Throughout this section we assume that t is even. We also assume that $p > 11$, so that G is neither PSL$(2,p)$ nor PGL$(2,p)$.

THEOREM 4.1. $\langle \chi^{(p-2,2)}, \chi^{(p-2,1^2)}\rangle_G = 0$.

Proof. I shall show that if χ is an irreducible character of G then χ cannot appear as a constituent both of $\chi^{(p-2,2)}$ and of $\chi^{(p-2,1^2)}$. Suppose first that $\chi(1) \neq p-1$. Then (Lemma 3.4) $\chi_{N(P)}$ has a linear constituent λ. Since b_0 is an involution, $\lambda(b_0)$ is 1 or -1. By Lemma 3.2, if $\lambda(b_0) = 1$

then λ does not appear in $\zeta_{N(P)}$, while if $\lambda(b_0) = -1$ then λ does not appear in $\eta_{N(P)}$. Consequently either $\langle \chi, \eta \rangle_G = 0$ or $\langle \chi, \zeta \rangle_G = 0$.

Suppose now that $\chi(1) = p-1$. Since $\chi^{(p-2,2)}$ and ζ are characters of projective modules every occurrence of χ in either involves an occurrence of a projective indecomposable which contains χ, and since $\chi^{(p-2,2)}$ and ζ have no linear constituents (Lemma 3.1), this projective indecomposable must be the inward link τ of χ. Consequently every occurrence of χ in $\chi^{(p-2,2)}$ or in ζ is paired with an occurrence of its non-linear mate φ. However $\varphi(1) \equiv 1 \bmod p$ since $\tau(1) \equiv 0 \bmod p$, and so we know already that φ cannot appear both in $\chi^{(p-2,2)}$ and in ζ. Therefore either $\langle \chi, \chi^{(p-2,2)} \rangle_G = 0$ or $\langle \chi, \zeta \rangle_G = 0$. Thus $\chi^{(p-2,2)}$ and ζ have no common irreducible constituents: and so $\langle \chi^{(p-2,2)}, \chi^{(p-2,1^2)} \rangle_G = 0$.

In **[14]**, Theorem 8.7, I have shown that this orthogonality relation is equivalent to little generosity of G.

COROLLARY 4.2. *If G is insoluble and $|N(P)|$ is even then G is a little generously 3-transitive.*

Just as in **[13]** there follow two more corollaries.

COROLLARY 4.3. *If G is insoluble and contains an odd permutation then G is a little generously 3-transitive.*

COROLLARY 4.4. *If X is 3-transitive of degree $p+2$ then either*

(i) *$p = 2^m - 1$ and X is $\mathrm{SL}(2, 2^m)$ or $\Sigma\mathrm{L}(2, 2^m)$ acting on the projective line over $\mathrm{GF}(2^m)$, or*

(ii) *X is 5-transitive and, for all $\Delta \subseteq \Omega$ with $|\Delta| = 6$, the set stabiliser $X_{\{\Delta\}}$ acts as $\mathrm{PGL}(2,5)$, A_6, or S_6 on Δ.*

The remainder of this section is devoted to the production of more detail about the characters $\chi^{(p-2,2)}$ and $\chi^{(p-2,1^2)}$, or equivalently about ξ. The argument in the proof of Theorem 4.1 shows already that if χ is an irreducible character of G and if $\chi_{N(P)}$ has linear constituents λ_1, λ_2 such that $\lambda_1(b_0) = 1$ and $\lambda_2(b_0) = -1$, then χ cannot appear in $\chi^{(p-2,2)}$ or in ζ, and hence not in ξ. Similarly, if χ has degree $p-1$ and if $\varphi_{N(P)}$ has linear constituents as above, where φ is the non-linear mate of χ, then χ cannot appear as a constituent of ξ.

LEMMA 4.5. *If χ is a p-rational irreducible character then $\langle \chi, \zeta \rangle_G \leqslant (p-1)/t$ and $\langle \chi, \chi^{(p-2,2)} \rangle_G \leqslant (p-1)/t$.*

Proof. To say that χ is p-rational and irreducible is merely to exclude the exceptional characters (if $t < p-1$). Therefore $\chi(1) = kp + \varepsilon$ where ε is 0, 1, or -1. If $\chi(1) \neq p-1$ then we know that $\chi_{N(P)}$ has a linear constituent λ. Since λ appears in ζ or in η with multiplicity $(p-1)/t$ we see that $\langle \chi, \eta \rangle_G \leqslant (p-1)/t$ and $\langle \chi, \zeta \rangle_G \leqslant (p-1)/t$, and the result follows. If $\chi(1) = p-1$, and if $\tau = \chi + \varphi$ is the inward link of χ, then we know that every appearance of χ in $\chi^{(p-2,2)}$, or in ζ, involves an occurrence of τ and hence of the non-linear mate φ. That is,

$$\langle \chi, \chi^{(p-2,2)} \rangle_G \leqslant \langle \varphi, \chi^{(p-2,2)} \rangle_G \leqslant (p-1)/t$$

and

$$\langle \chi, \zeta \rangle_G \leqslant \langle \varphi, \zeta \rangle_G \leqslant (p-1)/t.$$

This completes the proof in all cases.

Notice that in the case where $t = p-1$ it follows that, apart from χ_1, which appears in ξ with multiplicity 2, the character ξ is multiplicity-free. I would very much like to prove more in this case: that G is 4-fold transitive, which is equivalent to the irreducibility of $\chi^{(p-2,2)}$ and $\chi^{(p-2,1^2)}$.

Lemma 4.6. *If χ is one of the exceptional characters and $\langle \chi, \xi \rangle_G \geqslant 1$ then $\langle \chi, \xi \rangle_G = 1$. Moreover in this case $\langle 1, \chi \rangle_{N(P)} = 0$ and if λ is any non-principal linear character of $N(P)$ then $\langle \lambda, \chi \rangle_{N(P)} \leqslant 1$.*

Proof. Suppose that $\chi = \theta_j$ and that $\langle \chi, \xi \rangle_G = m$. Since ξ is rational-valued, while $\theta_1, \ldots, \theta_{(p-1)/t}$ are all p-conjugate, $\langle \theta_i, \xi \rangle_G = m$ for all i. Let λ be a linear character of $N(P)$ and suppose that $\langle \lambda, \theta_j \rangle_{N(P)} = l$. Then since λ is p-rational, $\langle \lambda, \theta_i \rangle_{N(P)} = l$ for all i. It follows that

$$(p-1)/t = \langle \lambda, \xi \rangle_{N(P)} \geqslant ((p-1)/t)lm.$$

By Lemma 3.4, $l \geqslant 1$ for some λ. Consequently $m \leqslant 1$. Moreover, if $l \geqslant 1$ for a character λ then $l = 1$. Finally, since χ would have to be a constituent of $\xi - 1$, while $\langle 1, \xi - 1 \rangle_{N(P)} = ((p-1)/t) - 1$, it follows that $l = 0$ if λ is the principal character of $N(P)$, that is, it follows that $\langle 1, \chi \rangle_{N(P)} = 0$.

Lemma 4.7. *If G has no subgroup of index 2, and if $t \equiv 0 \bmod 4$, then the exceptional characters do not appear in $\chi^{(p-2,1^2)}$.*

Proof. Let T be the 2-primary constituent of Q, and let b_1 be a generator of T. The order of b_1 is 2^m where $m \geqslant 2$. Further, since b_1 normalises P it has one fixed point and $(p-1)/2^m$ cycles of length 2^m in Ω; and as it must be an even permutation, else G would have a subgroup of index 2, it follows that $p-1 \equiv 0 \bmod 2^{m+1}$. Suppose that $\theta_i(1) = kp - \varepsilon t$ (as on p. 483). Then $\langle 1, \theta_i \rangle_P = k$ and so $(\theta_i)_{PT}$ has just k linear constituents. The non-linear irreducible characters of PT are characters induced from non-principal linear characters of P, and therefore, if φ is one of these, then $\varphi_T = \rho_T$, where ρ_T denotes the character of the regular representation of T. It follows that

$$(\theta_i)_T = r\rho_T + \sum_1^k \lambda_j,$$

where $r = (k(p-1)-\varepsilon t)/2^m$ and $\lambda_1, \ldots, \lambda_k$ are certain linear characters of T.

In the regular representation of T the generator b_1 is mapped to a matrix of determinant -1. Also, r is odd since $(p-1)/2^m$ is even and $t/2^m$ is odd; therefore in a sum of r copies of the regular representation b_1 has determinant -1.

Suppose now that θ_i is a constituent of $\chi^{(p-2,1^2)}$. We can suppose that $b_0 = b_1^{2^{m-1}}$, and from Lemma 3.2 it follows that $\lambda_i(b_0) = -1$ for $i = 1, 2, \ldots, k$. All the linear characters of T which satisfy this condition represent b_1 by a primitive 2^mth root of 1 and therefore are algebraically conjugate. Moreover, θ_i takes rational values on elements whose order is different from p. Consequently

$$\lambda_1 + \ldots + \lambda_k = (k/2^{m-1})(\lambda_1 + \ldots + \lambda_{2^{m-1}}),$$

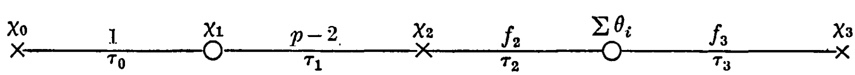
where the λ_i have been renumbered so that $\lambda_1, \ldots, \lambda_{2^{m-1}}$ form one algebraically conjugate set. Since the product of the 2^{m-1} primitive 2^mth roots of 1 is $+1$ if $m \geqslant 2$, the representation with character $\lambda_1 + \ldots + \lambda_{2^{m-1}}$ represents b_1 by a matrix of determinant $+1$. Therefore $\lambda_1 + \ldots + \lambda_k$ is the character of a representation in which b_1 has determinant $+1$ and so, finally, a representation with character θ_i represents b_1 with determinant -1. This contradicts the assumption that G has no subgroup of index 2, and shows that θ_i cannot be a constituent of $\chi^{(p-2,1^2)}$ after all.

EXAMPLE 4.8. Let p be a Fermat prime 2^m+1, and let G be $\mathrm{SL}(2, 2^m)$ acting in the natural way as a group of degree p. In this case $t = 2$ and $\theta_1 + \theta_2 + \ldots + \theta_{(p-1)/2}$ is mated with χ_1 in the projective indecomposable τ_1. This sum must appear therefore in $\chi^{(p-2,1^2)}$ and comparison of degrees shows that $\chi^{(p-2,1^2)} = \theta_1 + \theta_2 + \ldots + \theta_{(p-1)/2}$: though of course this can also be computed directly using the known values of these characters of G. This example shows that the condition $t \equiv 0 \bmod 4$ is essential to the lemma. Groups G such that $\mathrm{SL}(2, 2^m) \leqslant G \leqslant \Sigma\mathrm{L}(2, 2^m)$ show the condition that G have no subgroup of index 2 to be unavoidable also.

EXAMPLE 4.9: groups in which $t = 4$. Suppose that $t = 4$. If G has a subgroup H of index 2 then $N_H(P)$ must have order $2p$. Therefore by a theorem of Itô ([**11**], or [**10**], p. 614) $H \cong \mathrm{SL}(2, 2^m)$ where $p = 2^m+1$, and G is the semi-direct product $H\langle\sigma\rangle$ where σ induces the field automorphism of order 2. Consequently we assume that G has no subgroup of index 2.

It is conjectured that no such groups exist: and I have a warm feeling towards that conjecture since it was an unsuccessful attempt to settle it which led me to the results of [**13**] and the present work.

According to Lemma 4.7 the exceptional characters θ_i do not appear in $\chi^{(p-2,1^2)}$. In fact they do not appear in $\chi^{(p-2,2)}$ either. The argument is as follows. By Lemma 3.2, of the four linear characters of $N(P)$, there are just two, λ_0, λ_1, which appear in the restriction of $\chi^{(p-2,2)}$ to $N(P)$. The principal character λ_0 appears $\frac{1}{4}(p-1)-1$ times, and λ_1 has multiplicity $\frac{1}{4}(p-1)$. There are $\frac{1}{4}(p-1)$ characters θ_i: they all appear in $\chi^{(p-2,2)}$ with the same multiplicity; their restrictions to $N(P)$ all contain λ_0 with the same multiplicity; and their restrictions to $N(P)$ all contain λ_1 with the same multiplicity. It follows that if the characters θ_i appear in $\chi^{(p-2,2)}$ then $\theta_i(1) = kp-4\varepsilon$ where $k \leqslant 1$. A theorem of Feit ([**9**]) excludes all possibilities except perhaps that $\theta_i(1) = p+4$. Then, if χ_3 is an irreducible character (with $\chi_3(1) \equiv 1 \bmod p$) such that $\theta_1+\ldots+\theta_{(p-1)/4}+\chi_3$ is a projective indecomposable character τ_3 contained in $\chi^{(p-2,2)}$, then, since there must be a projective indecomposable τ_2 containing χ_2 and either χ_3 or $\sum\theta_i$, and since the mate of χ_2 in τ_2 has degree congruent to $-1 \bmod p$ the Brauer tree must be



In this diagram the number above each edge is the degree of the corresponding modular irreducible. Now

$$f_3 = \chi_3(1) \equiv 1 \mod p,$$

$$f_2+f_3 = \theta_i(1) = p+4,$$

and

$$p-2+f_2 = \chi_2(1) \equiv 1 \mod p.$$

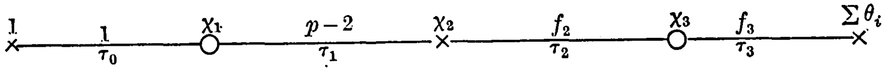
The only possibility is that

$$f_3 = p+1 \quad \text{and} \quad f_2 = 3.$$

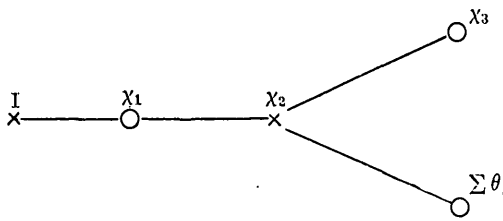
But we know that G cannot have a modular representation of degree 3 (see, for example, [8]). Thus we have the following result.

If $t = 4$ and G has no subgroup of index 2 then the exceptional characters do not occur in $\chi^{(p-2,2)}$ or in $\chi^{(p-2,1^2)}$.

We can go just a little further. Let χ_3 be the fourth p-rational character in $B_0(G)$. Then the Brauer tree is either as portrayed on p. 491, or it has the form



The only other possibility would be



and this is excluded by a result of Brauer ([**3**], p. 958) concerning the shape of the real stem. Since the exceptional characters do not appear, τ_3 cannot be a summand of $\chi^{(p-2,2)}$. Likewise, since χ_2 appears in $\chi^{(p-2,1^2)}$ it does not appear in $\chi^{(p-2,2)}$ and so τ_2 is not a summand of $\chi^{(p-2,2)}$ and we have shown that

if $t = 4$ then every irreducible constituent of $\chi^{(p-2,2)}$ has defect 0.

5. The decomposition of the permutation characters when t is odd

Throughout this section it is assumed that t is odd and that $p > 11$. We know already (Lemma 3.1) that the linear characters of G do not appear as constituents of $\chi^{(p-2,2)}$ or of $\chi^{(p-2,1^2)}$.

THEOREM 5.1. (i) *The exceptional characters do not appear as constituents of $\chi^{(p-2,2)}$ or $\chi^{(p-2,1^2)}$.*

(ii) *If χ is any irreducible character of G then*

$$\langle \chi, \chi^{(p-2,2)}\rangle_G \leqslant (p-1)/2t$$

and

$$\langle \chi, \zeta\rangle_G \leqslant (p-1)/2t.$$

N.B. This last inequality is equivalent to the statement that

$$\langle \chi, \chi^{(p-2,1^2)}\rangle_G \leqslant (p-1)/2t \quad \text{for} \quad \chi \neq \chi_1$$

and that

$$\langle \chi_1, \chi^{(p-2,1^2)}\rangle_G \leqslant ((p-1)/2t)-1.$$

Proof. (i) Since the exceptional characters $\theta_1, \ldots, \theta_{(p-1)/t}$ are all p-conjugate they appear in a rational-valued character such as η or ζ with the same multiplicity m. From Lemma 3.4 we know that $(\theta_1)_{N(P)}$ must have a linear constituent λ. As λ is p-rational it appears in

$(m\sum\theta_i)_{N(P)}$ with multiplicity at least $m((p-1)/t)$. But λ appears in $\eta_{N(P)}$ and in $\zeta_{N(P)}$ with multiplicity $(p-1)/2t$ (Lemma 3.3). Thus the characters θ_i cannot appear in η nor in ζ, therefore they cannot appear in $\chi^{(p-2,2)}$ nor in $\chi^{(p-2,1^2)}$.

(ii) Suppose that χ is irreducible and that $\chi_{N(P)}$ has a linear constituent λ. Then $\langle\lambda,\eta\rangle_{N(P)} = \langle\lambda,\zeta\rangle_{N(P)} = (p-1)/2t$. Therefore $\langle\chi,\eta\rangle_G \leqslant (p-1)/2t$ and $\langle\chi,\zeta\rangle_G \leqslant (p-1)/2t$. This argument applies to all irreducible characters of G except those of degree $p-1$. If χ has degree $p-1$ and φ is the non-linear mate of χ in the inward link $\tau = \chi+\varphi$, then every occurrence of χ in $\chi^{(p-2,2)}$ or in ζ involves a summand τ (because $\chi^{(p-2,2)}$ and ζ are characters of projective modules which have no linear constituents). Therefore

$$\langle\chi,\chi^{(p-2,2)}\rangle_G \leqslant \langle\varphi,\chi^{(p-2,2)}\rangle_G \leqslant (p-1)/2t$$

and

$$\langle\chi,\zeta\rangle_G \leqslant \langle\varphi,\zeta\rangle_G \leqslant (p-1)/2t,$$

and this completes the proof. As a corollary we have the following.

THEOREM 5.2. (i) *$G_{\alpha\beta}$ has at most $(p-1)/t$ orbits in $\Omega\setminus\{\alpha,\beta\}$.*
(ii) *$G_{\{\alpha\beta\}}$ has at most $1+(p-1)/2t$ orbits in $\Omega\setminus\{\alpha,\beta\}$.*

Proof. (i) The permutation character of G on $\Omega^{(2)}$ is ξ, and on Ω is $\tau_0 = 1+\chi_1$. We have from Theorem 5.1 that

$$\begin{aligned}\langle\xi,\tau_0\rangle_G &= 1+\langle\chi_1,\chi_1\rangle_G+\langle\chi^{(p-2,2)},\chi_1\rangle_G+\langle\zeta,\chi_1\rangle_G\\ &\leqslant 2+(p-1)/t.\end{aligned}$$

This means that G has at most $2+(p-1)/t$ orbits in $\Omega^{(2)}\times\Omega$, or that $G_{\alpha\beta}$ has at most $(p-1)/t$ orbits in $\Omega\setminus\{\alpha,\beta\}$.

(ii) Similarly, $\langle\eta,\tau_0\rangle_G \leqslant 2+(p-1)/2t$ and so a stabiliser $G_{\{\alpha\beta\}}$ for $\Omega^{(2)}$ has at most $2+(p-1)/2t$ orbits in Ω, therefore at most $1+(p-1)/2t$ orbits in $\Omega\setminus\{\alpha,\beta\}$.

An application of Theorem 5.2 has been given by M. D. Atkinson in [**1**]. Another, to the case where $t=(p-1)/2$, is the subject of a sequel to this paper [**15**].

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